Aggregative Environmental Games

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Abstract

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1 Introduction

At the heart of the subdiscipline of environmental economics is a judgement, shared by this author, that even in the absence of the other standard sources of "market failure", a decentralized market equilibrium is likely to be inefficient as a consequence of various externalities, or spillovers, that are generated by production and consumption activities. These links, that are not mediated through the system of voluntary exchange, and may be variously categorized as pollution, congestion, resource depletion, over-exploitation, have quantitatively significant implications for human welfare. The tasks of environmental economists include the attempts to quantify these externalities, explore their role in generating inefficient equilibria, and to suggest implementable mechanisms that overcome, or at least mitigate, the inefficiency.

This paper is concerned with the issue of how best to analyze situations involving reciprocal externalities. I consider two basic models, and various extensions of those models. The first – the voluntary pure public good contribution model – captures the essential features of situations involving positive, or beneficial, reciprocal externalities. The second – the open access resource model, or the tragedy of the commons - captures the essential features of situations involving negative, or detrimental, reciprocal externalities.

Using these two simple models as my starting point, I want to suggest a novel method of analyzing them that avoids inherent limitations associated with the use of best response functions. In particular, I suggest two tricks that both simplify, and greatly extend the power of, the analysis of such situations. In the context of the pure public good model, I introduce the use of 'replacement functions' for describing individual players' behavior. In the context of open access resources, I explore a slight variant that uses 'share functions'. Both tricks depend upon the game having a certain aggregative structure. By exploiting that structure, they avoid what Richard Bellman, in another context, called the curse of dimensionality. I show how these approaches provide a simple, graphically transparent, and powerful way of tackling the issues of existence, uniqueness and comparative static analysis of reciprocal externality models. The resulting treatment remains analytically tractable even in the presence of a large number of heterogeneous players.

In addition, I consider a number of extensions or adaptations of the basic models, and show how the suggested alternative approach remains relevant and helpful. Throughout, I try to indicate actual or potential applications of these ideas to issues that particularly concern environmental economists.

To keep the discussion short and transparent, I shall not provide technically rigorous proofs of many of the claims made below. Interested readers can find these in the various cited references.

2 Pure Public Goods and Replacement Functions

2.1 The Basic Pure Public Good Model[1]

Our basic model of pure public goods envisages n players. Player $i, i = 1, ..., n$, has preferences represented by a utility function $u_i = u_i(x_i, Q)$. The objects x_i and Q are, respectively, the quantities of a single private good and of a public good.

- The utility function $u_i = u_i(x_i, Q)$ is everywhere differentiable, strictly increasing and strictly quasiconcave in its two arguments.

- Player i has a linear budget constraint of the form

  $$x_i + q_i \leq m_i$$

  where $q_i \geq 0$ is her contribution to a pure public good. Income m_i is strictly positive and exogenous.

- The 'Social Composition Function' that determines the aggregate quantity Q as a function of individual contributions takes a simple summation form:

  $$Q = \sum_{j=1} q_j = q_i + Q_{-i}$$

  where Q_{-i} is the sum of the contributions made by all players except i.

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[1] The exposition of this model draws on Cornes and Hartley [9], which provides a fuller and more rigorous account.

The model is a noncooperative game in which each player is choosing her contribution so as to maximize her utility, with the contributions of all other players treated parametrically.

Finally, I assume that, for every player i, both x_i and Q are normal. That is, the locus of values of x_i and Q consistent with a given marginal rate of substitution has positive finite slope everywhere.

Player i chooses nonnegative values of x_i and q_i to maximize utility subject to her budget constraint and the prevailing value of Q_{-i}. To any non-negative value of Q_{-i} there corresponds a unique utility-maximizing contribution level, $\hat{q}_i$. By varying Q_{-i} parametrically, we generate her best response function, $\hat{q}_i = b_i(Q_{-i})$. At a Nash equilibrium, every player's choice is a best response to the prevailing choices of all other players. The typical player's preferences and constraint set are depicted in Figure 1(a). To a microeconomist, the model is one of straight rationing, in which each player enjoys a given ration Q_{-i} of some good, but can choose to augment this by choosing to consume, or contribute, the quantity q_i of a good that is a perfect substitute for the rationed quantity.

Under these assumptions, each player's budget constraint will, in fact, hold with equality at equilibrium. This observation allows us to rewrite her utility function as $\nu_i(q_i, Q : m_i) = u_i(m_i - q_i, Q)$. This is the objective function from which I will start in my derivation of individual behavior.

2.2 Individual Replacement Functions

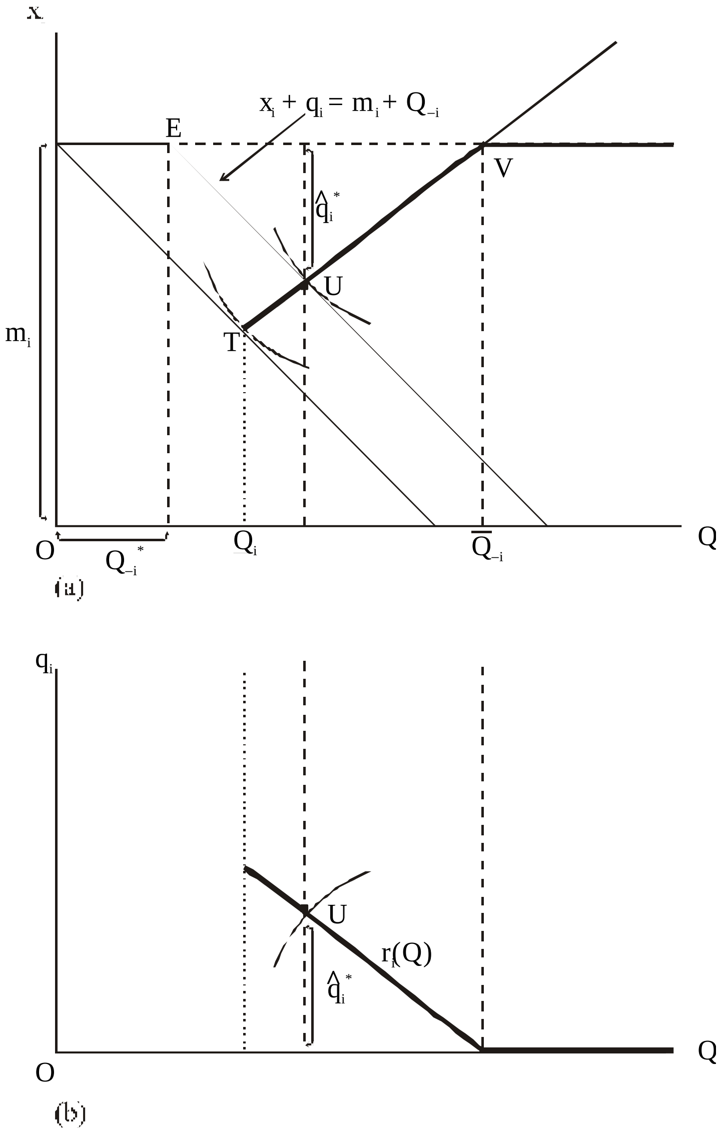
For the moment, suppose that player i's best response is at an interior solution, at which $0 < \hat{q}_i < m_i$. Such a solution must satisfy the first-order condition for utility maximization:

$$\frac{\partial \nu_i(q_i, Q : m_i)}{\partial q_i} + \frac{\partial \nu_i(q_i, Q : m_i)}{\partial Q}\frac{\partial Q}{\partial q_i} = 0.$$

Using the definition of Q, and the fact that the contributions of others are being treated parametrically, this can be written as

$$\phi_i(\hat{q}_i, Q : m_i) = \frac{\partial \nu_i(\hat{q}_i, Q : m_i)}{\partial q_i} + \frac{\partial \nu_i(\hat{q}_i, Q : m_i)}{\partial Q} = 0.$$

This is an implicit function in $\hat{q}_i$ and Q. Graphically, it describes the points on the strictly upward-sloping portion, TV, of the individual's income expansion path in Figure 1(a). For example, given a value Q^*_{-i}, the player's endowment point is the point E, her budget line is the line with slope -1 through E and her chosen allocation is the tangency at an indifference curve at U. Her contribution to the public good is the vertical distance between theAggEnvGames-fig1.WMFne corresponding to her given money income. I have drawn the figure on the assumption that if Q_{-i} equals or exceeds some finite value, $\overline{Q}_{-i}$, then player i spends all her income on the private good. The figure suggests, and the analysis of Cornes and Hartley confirms more rigorously, that there is a unique value of best response $\hat{q}_i$ consistent with any value of the aggregate


x
$x_i + q_i = m_i + Q_{-i}$
E
$\hat{q}_i^*$
U
V
m_i
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Q_i
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Q
(a)
q_i
U
$r_i(Q)$
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(b)

Figure 1:

$Q \in \left[\underline{Q}_i, \infty\right)$. The value is the level of player i's chosen contribution in a situation when no other players are making a positive contribution. I shall write the explicit function that uniquely determines $\hat{q}_i$ for any observable value of Q as

$$\hat{q}_i = r_i\left(Q, m_i\right)$$

and I shall call $r_i\left(.\right)$ player i's replacement function[2]. The graph of $r_i\left(Q, m_i\right)$ is depicted in Figure 1(b).

The assumptions of this model imply the following properties of individual replacement functions: For all i,

1. There exists a finite value, $\underline{Q}_i$, at which $r_i\left(\underline{Q}_i, m_i\right) = \underline{Q}_i$,
2. $r_i\left(\underline{Q}_i, m_i\right)$ is defined for all $Q = \underline{Q}_i$,
3. $r_i\left(\underline{Q}_i, m_i\right)$ is continuous,
4. $r_i\left(\underline{Q}_i, m_i\right)$ is everywhere nonincreasing, and is strictly decreasing wherever it is strictly positive.

2.3 Equilibrium and the Aggregate Replacement Function

First, define the aggregate replacement function of the game, $R(Q)$, as

$$R\left(Q, \mathbf{m}\right) = \sum_{j=1} r_j\left(Q, m_j\right)$$

where $\mathbf{m} \equiv \left(m_1, m_2, ..., m_n\right)$. Now note that a Nash equilibrium is simply an allocation at which the aggregate value, Q, equals the sum of individual best responses - or replacement values - that are consistent with it:

$$R\left(Q, m\right) = \sum_{j=1} r_j\left(Q, m_j\right) = Q. \tag{1}$$

By contrast with the best response approach, which generates n equations in n unknowns, this remains a single equation in a single unknown, regardless of how many heterogeneous players are in the game. Moreover, it is a simple matter to show that a unique solution to this equation exists. The analysis uses the following properties of $R(.)$, which are inherited in a straightforward manner from those of the individual replacement functions:

1. $R\left(\max\left\{\underline{Q}_1, \underline{Q}_2, ..., \underline{Q}_n\right\}, \mathbf{m}\right) = \max\left\{\underline{Q}_1, \underline{Q}_2, ..., \underline{Q}_n\right\}$

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[2] The rationale for this name is as follows: Given any observable value of the aggregate, Q, there is a uniquely determined quantity - say Z - that can be subtracted from Q such that player i's best response to $(Q - Z)$ is precisely the quantity Z. In other words, player i's best response exactly replaces the quantity Z.

2. $R(Q, m)$ is defined for all $Q = \max \left\{ \underline{Q}_1, \underline{Q}_2, \ldots, \underline{Q}_n \right\}$
3. $R(Q, m)$ is continuous.
4. $R(Q, m)$ is everywhere nonincreasing in Q, and is strictly decreasing wherever it is strictly positive.

Figure 2 depicts the graphs of the individual replacement functions, and of $R(Q, m)$, in a 4-player model. In the example shown, the point N depicts the sole equilibrium allocation. At that equilibrium, players 3 and 4 are choosing to make strictly positive contributions, while players 1 and 2 are enjoying completely free rides.

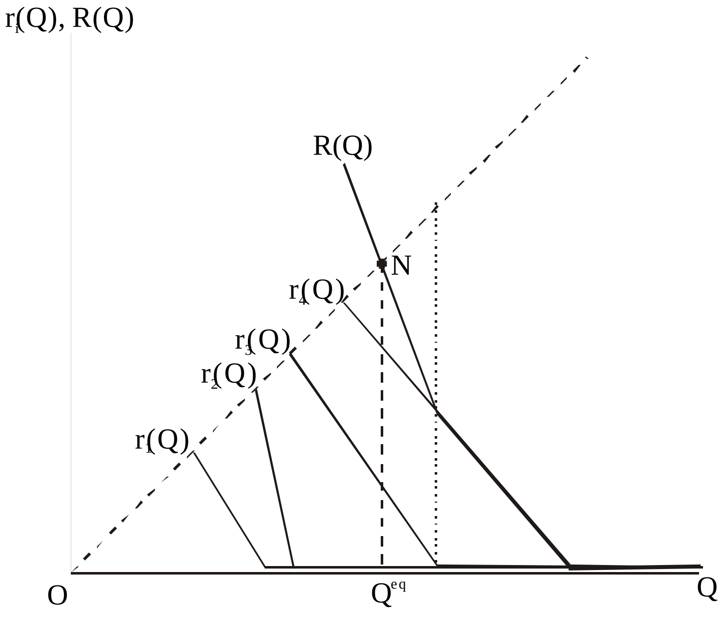

Figure 2:

2.4 Some Comparative Statics

Our approach provides a simple and direct method of comparative static analysis. We confine attention here to two celebrated properties of the model, and invite the interested reader to consult Cornes and Hartley [9] for further discussion.

First, consider a population of individuals consisting of a finite number, N, of types. Type r individuals cease to be positive contributors at the point where $Q = Q_r^o$. We adopt the convention that $Q_1^o < Q_2^o < \ldots < Q_N^o$. Now consider the behavior of the equilibrium of a public good model as individuals are randomly drawn from this population and added to the economy. As the number of draws becomes large, so will the number of type N players who are drawn. It is easy to understand that, at some point, the aggregate replacement function of type N players will become so steep that the equilibrium value of Q exceeds the value at which any of the other types is willing to make a positive contribution. In short, we may expect a large economy to be at an equilibrium at which a very small proportion of its members are making positive contributions. This was pointed out explicitly by Andreoni [1], and has provided ammunition for critics of this simple model, who have argued that this prediction often fails to be borne out in situations to which this model has been applied.

The second property of the model to which we want to draw attention is the well-known neutrality property. Look again at Figure 1. Suppose the player receives an extra unit of money income. This shifts the money income line up by one unit All other features of Figure 1(a) remain unchanged. In Figure 1(b), the interior section of the graph of the replacement function shifts up by one unit.

Now consider an equilibrium in which players j and k, whose money incomes are. respectively m_j and m_k, are positive contributors. The neutrality property concerns the comparative static consequences of a transfer of income from j to k, so that their initial money incomes, instead of being m_j and m_k, become $m_j - \Delta$ and $m_k + \Delta$. In the neighborhood of the initial equilibrium, player j's replacement graph shifts down by the amount , and player k's shift upwards[3]. The aggregate replacement graph therefore remains unchanged, and the equilibrium is undisturbed by the initial income transfer. This observation has a natural extension to many players – transfers of income amongst a set of positive contributors that do not push any one of them to a corner solution have no effects on the equilibrium outcome. This dramatic prediction again has provided a target for criticism of the model as being in conflict with observed behavior in some contexts to which it has been applied.

3 Pure Public Goods: Extensions and Applications

The sharp predictions of the pure public good have long fascinated theorists. However, as we have just pointed out, they have also provided ready targets for more empirically minded critics. As a model of charitable giving, it has been criticized on the grounds that the neutrality proposition does not seem to hold. Furthermore, the prediction that, in an economy with many potential contributors, a very small proportion will make positive contributions in equilibrium, has also cast doubt on its plausibility. For these and other reasons, various extensions have been suggested. I will explore two avenues that have been developed. One modifies players' preferences. The other considers more general

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[3] We are assuming that the transfer, and therefore the shift in player j's graph, is not large enough to meet the individual nonnegativity constraint on $\widehat{q}_j$.

processes by which individual contributions are transformed into the aggregate Q.

3.1 Impure, or Joint Product, Public Good Models

Cornes and Sandler [12] considered a model in which the consumption of a particular commodity jointly generates both a private and also an addition to the aggregate quantity of a public characteristic[4]. Player i's preferences over three characteristics are represented by the function

$$u_i\left(x_i, y_i, Z\right). \tag{2}$$

Her budget constraint is

$$x_i + pq_i = m_i \tag{3}$$

The quantity x_i is a numeraire, and the quantity q_i is a marketed commodity that jointly generates consumed characteristics according to the following relationships:

$$\begin{aligned} y_i &= \beta q_i & (4) \\ Z &= Z_{-i} + \gamma q_i & (5) \end{aligned}$$

where Z_{-i} is the level of the public characteristic generated by the consumption of others, p is the price of the good q_i, and β and γ are exogenous parameters. Cornes and Sandler suggested the possibility of modeling charitable giving in this way, and Bergstrom, Blume and Varian suggested the interpretation of the jointly produced private characteristic as reflecting the 'warm glow' that contributors feel as a result of 'doing their bit'. This particular application has been further discussed, tested empirically and criticized by many others. However, I want to emphasize that the present formal extension has a much broader range of applications. This is evident when one observes that we can allow every player to regard the aggregate Z as a bad, for which $\partial u_i(.)/\partial Z$ is everywhere negative for all i. Even in this situation, every player may be given an incentive, through their desire for the jointly produced private characteristic, to generate the bad, Z. This immediately suggests that this model should interest environmental economists, since Z may be interpreted as environmental pollution or congestion produced as a by-product of consumption of the private characteristic y.

I have already noted that, formally, the pure public good model is one of straight rationing. The present impure public good model may be regarded as one of points rationing in the space of the characteristics (x_i, y_i, Z), in which the individual player maximizes her utility subject to two resource constraints. This may be seen by using and to rewrite the constraints in terms of the nontraded characteristics that directly generate utility. The problem becomes

$$Maximise_{x_i, y_i, Z} \quad u_i\left(x_i, y_i, Z\right)$$

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[4] See also Cornes and Sandler [14] for further development of this model.

$$\text{s.t.} \quad x_i + \left(\frac{p}{\beta}\right) y_i = m \tag{6}$$

$$-\gamma y_i + \beta Z = \beta Z_{-i} \tag{7}$$

Figure 3 depicts the two linear constraints in (x_i, y_i, Z) space. The constraint (6) is depicted by the plane ABCD, and (7) is depicted by the plane TUV.

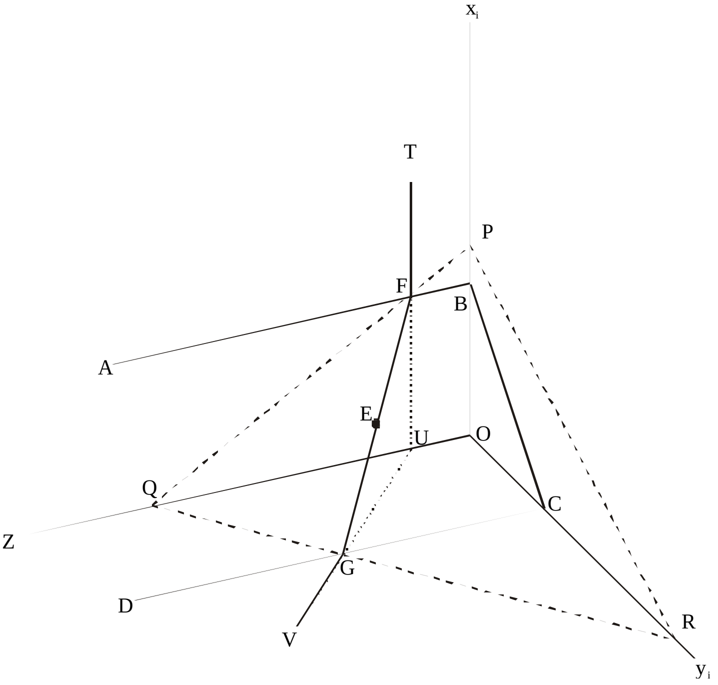

Figure 3:

Under our assumptions, the most preferred point for an individual faced with these constraints will be somewhere on the intersection, FG, of these two planes. Suppose it is at the point E where an indifference curve touches the line segment FG.

The richer behavior consistent with this impure public good model may be appreciated by considering the consequences for a player's best response of an exogenous increase in money income. Other things held equal, in the pure public good model, this will increase her contribution to the public good. But in the present model, even if the three nontraded characteristics are all normal goods in the usual sense, this need not be true.

The increase in money income shifts the plane ABCD in Figure 3 outward, in a parallel fashion, away from the origin. The point of tangency between an indifference curve and the intersection of the two constraint planes will shift. But the player is not facing a parallel shift in a linear plane in 3 dimensions.

Although the plane ABCD has shifted, the new point of tangency must remain in the plane TUV. This forces the change in consumption y_i and that of the contribution by player i to the good Z to be in the proportions determined by the exogenous β and γ parameters: $\frac{dy_i}{dZ} = \frac{\gamma}{\beta}$.

In order to gain insight into the comparative static changes provoked by an increase in m_i, Cornes and Sandler exploit the idea of virtual prices and income. There is a budget plane in $(x_i, y_i.Z)$ space,described by a set of virtual prices and virtual income $[\pi_x(= 1), \pi_y, \pi_Z, \mu]$, that would support the chosen point E as a utility-maximizing allocation[5] by a price-taker in the characteristics space. This is shown by the plane PQR in Figure 3. Suppose that a player, starting at E, enjoys a money income increase. The points rationing model forces her to increase y_i and Z to maintain the equality $\beta dy_i = \gamma dZ$. The new intersection of the two planes is the line F^{0}G^0 in Figure 4. Although this intersection has

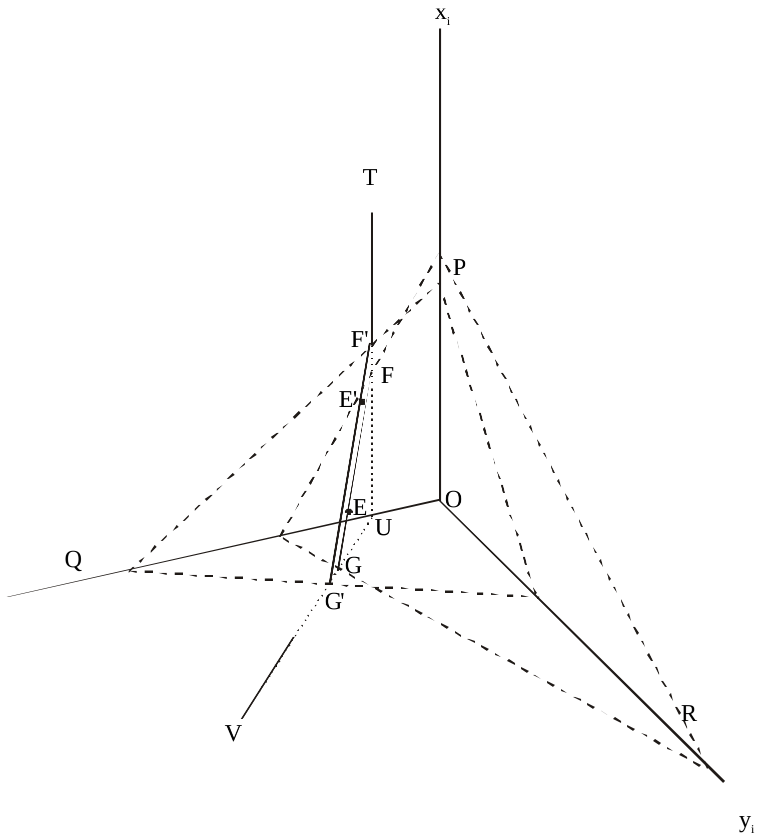

Figure 4:

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[5] By putting $\pi_x = 1$ throughout, we are adopting characteristic x as the numeraire, in terms of which quantities are being measured in the virtual price regime.

shifted in a parallel fashion, the shape of the player's indifference map in the neighborhood of the initially chosen point E may be such that the new point of tangency is at a point such as E' in Figure 4. I have also shown the implied virtual budget constraint that would support the choice of E'. This is the plane $P'Q'R'$. As shown, the individual has experienced a fall in income in terms of characteristic x, and a significant increase in the price of the public characteristic, Z. This explains why it is possible that, in the shift to E', the level of the player's contribution to the public characteristic may fall, along with her consumption of characteristic y.

Significantly, although this framework has been appealed to in a number of applications, little explicit attention has been given to the issues of existence and uniqueness of equilibrium. These issues are not addressed by Cornes and Sandler or by Ruebbelke, who provides a lengthy discussion of the model's application to global warming. However, Kotchen [19] has recently used the replacement function approach to provide such an analysis. To keep matters simple, he puts $p = \beta = \gamma = 1$. This is merely a normalizing exercise, and it allows us to write the player's utility function as $u_i(x_i, q_i, Q)$ Denoting the player i's most preferred level of the public characteristic by $f_i(m_i + Q_{-i}, Q_{-i}) \geq Q_{-i}$, her best response function may be written as

$$\hat{q}_i = f_i(m_i + Q_{-i}, Q_{-i}) - Q_{-i} \geq 0,$$

Kotchen points out that, if the following 'normality' assumption is assumed:

$$0 < \frac{\partial f_i(.)}{\partial Q_{-i}} < 1,$$

then there exists a replacement function, $r_i(Q)$, which is continuous at all $Q \geq \underline{Q}_i$, and which satisfies $r_i\left(\underline{Q}_i\right) = \underline{Q}_i$ and $r_i'(Q) \leq 0$. In short, for a given level of money income the replacement function has the same salient properties as did the replacement function in the pure public good model. Existence and uniqueness of equilibrium immediately follow.

3.2 Applications and extensions of the impure public good framework

A number of authors have explored further the comparative static properties of the impure public good model and of slight variants thereof.

Cornes and Sandler explore further the comparative static implications of a new technology becoming available with a higher value of either β or γ. They show that a small increase in either β or γ alone can lead to an equilibrium which is Pareto-dominated by the initial equilibrium. The individual player will choose the technology associated with the higher value of β [or γ], but may choose a point at which her contribution to the public good is smaller than initially. If all players respond in this way, it is possible for the new, unambiguously more productive, technology to generate an equilibrium in which all are worse off than at the intitial equilibrium. They also consider a situation in which each player can choose between two technologies, characterized respectively by the parameter values (β, γ), and $\left(\beta', \gamma'\right)$, where $\beta < \beta'$ and $\gamma > \gamma'$.Neither technology is unambiguously the more productive. They show that it is possible that all players choose the technology (β, γ) even though the technology $\left(\beta', \gamma'\right)$, if chosen by all, leads to a Pareto-superior equilibrium. The converse is also possible - all may choose the technology $\left(\beta', \gamma'\right)$ even though the technology (β, γ) would lead to a Pareto-superior equilibrium. It all depends on the magnitudes of the income and substitution parameters in characteristics space[6].

Ruebbelke [21], [22] has applied the impure public good model to the analysis of global warming. In his analysis, the players are governments. A policy of, say, acid rain or greenhouse gas abatement not only generates a regional public good, but also generates ancillary benefits that accrue particularly to its own citizens. He exploits the comparative static analysis of Cornes and Sandler to explore the complex interrelationship between countries' environmental policies.

More recently, Kotchen has used a similar framework to analyze other environmental issues. Motivated by the recent growth of interest in environmentally friendly consumption, and of fair trade goods, he treats such goods as embodying both a private characteristic - , for example, the tasty coffee - and the public characteristic - the contribution to enhancing the livelihood of the women of Peru who cultivate, harvest and process the coffee beans[7]. In Kotchen's framework, it is possible to buy this joint product, or to buy the private characteristic alone - ordinary coffee. He considers whether the availability of the green, or fair trade, good necessarily leads to an increased provision of the public good and shows that the answer is not straightforward. Among other things, he shows that

> ...increased demand for a green product or imp[rovements in a green product's technology can have detrimental effects on environmental quality. Kotchen [17],p.298].

In subsequent work, Kotchen has taken the analysis of equilibrium in this model further - see Kotchen [18]. Figure 5 summarizes the relationship between the earleir model of Cornes and Sandler, and Kotchen's later setup. In panel (a), the individual chooses beteen the numeraire private good and the activty that

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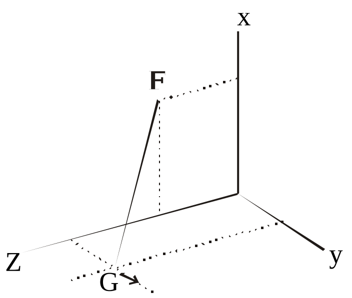
[6] It should be confessed that, with the benefit of hindsight, their analysis was a little inelegant, and failed to recommend itseld to referees or editors.

[7] This reference is inspired by the following excerpt from a brochure for Café Femenino, which the author enjoys drinking:

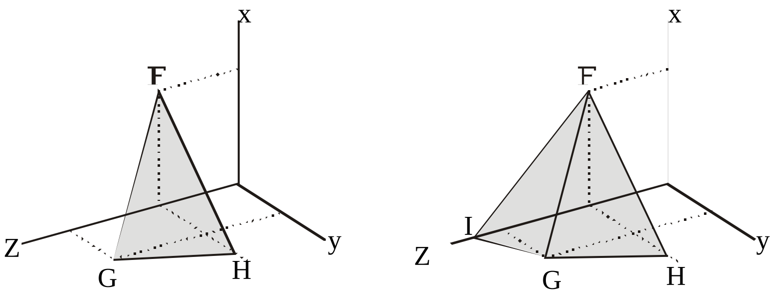
" Despite numerous advances in many parts of the world, the marginalisation and oppression of women still occurs. Especially in areas of extreme poverty, such as the rural communities of Peru.

But now with Café Femenino Fairtrade & Organic premiums, the dreams of female coffee farmers is a reality. In a World First, this coffee from the Penachi region in Peru is produced by women only. This business venture is the beginning of breaking the cycle of abuse and poverty by providing direct income to the women and raising their self esteem".

[Brochure, Jasper Coffee]



(a) Cornes and Sandler (EJ 1984)



(b) Kotchen (JEEM 2005)

(c) Kotchen (JEEM 2005)

Figure 5:

jointly generates a public and a second private characteristic. In panel (b), the individual may in addition choose the second private characteristic - ordinary coffee - without jointly adding to the public characteristic. If she can purchase both ordinary and fairtrade coffee, her constraint set is, effcetively, the plane FGH. If, in addition, she is able to make donations directly to the women of Peru without buying their coffee, her constraint set also includes triangle FGI, shown in panel (c).

The impure public good model, though analytically tractable, is sufficiently intricate to stretch the intuition of the analyst. For this reason it is worth considering special cases that accommodate its insights but are a little easier to work with. Vicary [24] uses the replacement function to analyze such a model. He is interested in situations where the public good takes the form of an action to mitigate the level of some public bad such as global warming. This mitigation takes place, not through subscription to some good, but through abstinence from consuming those goods and services whose consumption or producton pollutes the environment. He envisages a set of N countries. There are n_i individuals in country i. Each individual has identical preferences over two goods represented by a utility function, $U(z, G)$. The quantity z is total consumption, and is determined as a linear combination of two tradable commodities: $z = ax + by$, where a and b are parameters, and x and y are the two tradable commodities. Commodity x is environmentally neutral, and commodity y is harmful, in the sense that higher values of y lead to lower levels of the public good G. In the absence of human activity, there is what Vicary calls a "Garden of Eden" level of provision of the public good, denoted by G^*. This measures the quality of the pristine environment. The actual level of G enjoyed by the players is given by $G^* - \sum_{i=1}^{N} \beta n_i y_i$, where β is a parameter. We cannot go into the details of Vicary's modeling, but for present purposes two points are notable about his analysis. The first is that his framework simplifies the Cornes-Sandler model by effectively assuming the two commodities, x and y, are perfect substitutes [by virtue of the fact that it is their weighted sum, z, that the individual cares about]. Thus, analysis can proceed, from the outset, in a 2-dimensional consumption space. Second, he uses the idea of the replacement function to condition the damage done by the individuals in country i, $r_i(D, n_i)$ on the total environmental damage $D = \sum_{i=1}^{N} D_i = \sum_{i=1}^{N} \beta n_i y_i$. An interesting feature of the replacement function in this model is that it is bounded above, and in general has a horizontal section. Moreover, if a country experiences an increases in its income, the horizontal section of the graph of its replacement function rises, but the downward-sloping section falls to the left. These features are shown

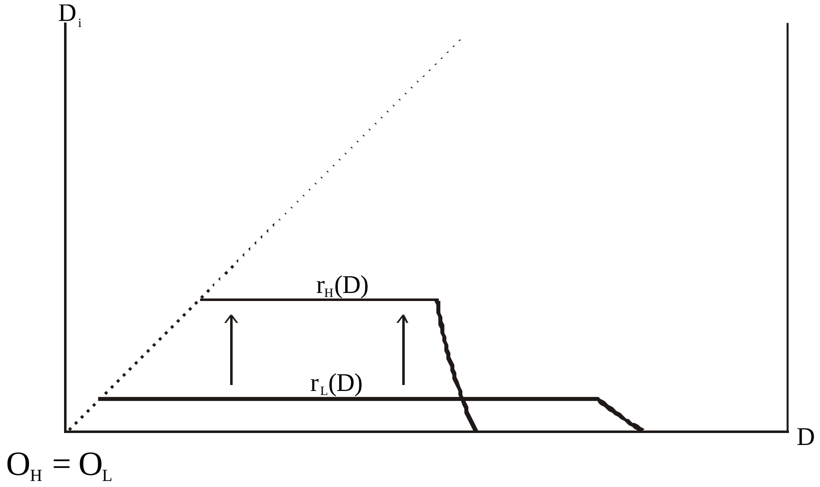


Figure 6:

in Figure 6, which shows the replacement functions of two countries that are identical except that one has a higher income than the other. There are, in an obvious sense, two opposing effects of income on the replacement function, and Vicary shows that this feature can provide a parsimonious explanation of the ‘[inverse] environmental Kuznets curve’ that some claim to find in data sets.

3.3 Idiosyncratic Productivities in the Public Good Model

We return to the pure public good model.The basic model is one of simple subscription, in which it is as if dollars are put into a hat, or collection box, and then added up and devoted to acquiring the public good. From a production theoretic viewpoint, the technology that transforms the individual contributions into the aggregate is a simple Ricardian one with perfectly substitutable inputs. Recent concern with regional and global public goods has provoked interest in developing more general models whereby a public good can be produced by individual efforts. Ihori [16] explored the implications of assuming that each player has her own productivity, or unit cost, as a generator of the public good. This is a natural first step in moving away from a simple subscription model of public good provision towards situations in which their generation is thought of as a production activity. This extension takes the standard basic model as described above, except that player i’s resource constraint is now

$$x_i + c_i q_i = m_i, \quad i = 1, 2, ..., n,$$

where c_i is player i’s exogenous unit cost parameter.

This has some significant, and surprising, implications. Suppose players j and k are both making positive contributions at the equilibrium of an n-player contribution game. Let them have identical preferences. Suppose that $c_j > c_k$. Then, at the equilibrium, player k, who has the lower unit cost as a public good provider, also will have a lower consumption of the private good and, therefore, a lower utility level. It does not pay to be a more efficient generator of the public good, since the gains of your higher productivity are more than fully transferred to the other members of the economy, who benefit from your higher contribution. Furthermore, if you take steps to enhance your productivity as a public good producer, it may be shown that, in equilibrium, this will certainly make all other players better off. But you may be made either better or worse off. The model suggests that the individual incentives to improve the technology of public good provision may be rather weak.

A last implication to which we draw attention here concerns the implication of income transfers between individuals with different unit costs. Retaining the assumption that $c_j > c_k$, a transfer from j to k will shift k’s replacement graph upwards by more than it shifts j’s graph downwards in the neighborhood of the initial equilibrium. As a consequence, the equilibrium level of Q must rise. Since both players have moved upwards along their income expansion paths, the result is a Pareto improvement over the initial equilibrium[8].

3.4 Weak-link and Good-shot Public Goods

The extension explored in the previous section is a minimal one. The production-theoretic approach suggests a more far-reaching extension in which the process

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[8] Cornes and Hartley [9] provide a more detailed discussion of these issues.

whereby individual contributions generate the aggregate quantity Q is a more general production process. Cornes [6] examines the class of models in which it is described by a CES production function:

$$Q = \alpha \left[\frac{1}{n} \sum_{j=1} q_j^{\eta} \right]^{\frac{1}{\eta}}$$

where $\alpha > 0$ and $\eta \neq 0$ are parameters. The standard summation case is obtained by putting $\alpha = n$ and $\eta = 1$.The assumption that $\eta < 1$ captures the idea that, at any allocation, if player i's contribution exceeds that of player j, then at the margin player i's is less productive. This is the case of the weaker link. One might plausibly argue that if one or two farmers in a region devote little attention to controlling weeds in their fields, this reduces the marginal effectiveness of weed control measures by others. Burnett [4] has recently analyzed the prevention of biological invasions - for example, the maintenance of Hawaii's mosquito-free status - as an example of a weaker link public good. The assumption that $\eta > 1$ corresponds to settings in which it is the higher individual contributions that have the higher marginal productivities. These are better shot situations.

Both Cornes and Burnett focus on the special case of the 2-player model with Cobb-Douglas social composition function in their formal analyses. This is simply because, if one works with best response functions, more general social composition functions and increasing the number of heterogeneous players make the model intractable. In this section, I sketch how the present approach may be exploited to overcome this problem.

Recall that the equations of the model are

$$\begin{array}{rl} \text{Preferences:} & u_i = u_i\left(x_i, Q\right) \\ \text{Budget constraints:} & x_i + q_i = m_i \\ \text{Social composition function} \ : & Q = \alpha \left[\frac{1}{n} \sum_{j=1} q_j^{\eta} \right]^{\frac{1}{\eta}} . \end{array}$$

Introduce a new set of variables, defined as

$$\begin{aligned} z_i &= q_i^{\eta}, \ i = 1, 2, ..., n \\ \text{and } Z &= \sum_{j=1} z_j. \end{aligned}$$

Player i's utility function may be written in terms of the new variables as

$$\begin{aligned} u_i &= u_i \left(m_i - z_i^{\frac{1}{\eta}}, \alpha \left(\frac{1}{n} Z \right)^{\frac{1}{\eta}} \right) \\ &= \nu_i \left(z_i, Z\right) \end{aligned}$$

Clearly, we now have a game that is aggregative. We can therefore use our approach to analyze the equilibrium of the game in the transformed variables. Having done so, we can back out the implied equilibrium values of the original variables. The trick used here is of more general use. Cornes and Hartley [11] show that any game in which each players payoff function is a function of their own choice and of a common additively separable function of all players' choices can be turned into an simple aggregative game by a transformation of variables[9].

3.5 Weakest Links and Best Shots

The analysis of Cornes was preceded, and indeed inspired, by the discussion of two extreme special cases by Hirshleifer [15]. These were the weakest-link and best-shot models, in which the generation of the total public good by individual contributions is described as follows:

$$\text{Weakest link:} \qquad Q = \min\{q_1, q_2, ..., q_n\}$$
$$\text{Best shot:} \qquad Q = \max\{q_1, q_2, ..., q_n\}.$$

In all other respects, the basic model is unchanged. The weakest link, in which the total quantity of the public good equals the smallest individual contribution, is familiar to production theorists as describing a Leontief technology. Hirshleifer's main example of a weakest link situation involves the perfectly circular island of Anarchia, of which each citizen owns a wedge-shaped piece. Each is responsible for the maintenance of her section of the circular dike surrounding the island, which protects it from occasional storms. If rough seas penetrate the dyke, they do so at the weakest - that is, least well maintained - section, and cause equal damage throughout the island. Thus the level of security enjoyed by each citizen depends, not on the sum of their maintenance levels, but on the least well maintained stretch of dike. This is the weakest link.

These two extreme cases have recently been analyzed in Cornes and Hartley [10] as limiting cases of a CES specification. In this paper, we will concentrate on analyzing the two extreme cases model in their own right, and show how our approach may be modified. The weakest link has been argued to characterize a number of issues that should be close to an environmental economist's heart. Sandler [23] suggests a number of such applications.

3.6 Representing Individual Behaviour and Equilibrium in the Weakest-link Model

3.6.1 Individual Behaviour

Again, we take the behaviour of other players as parametrically given. For a given value of the smallest individual contribution by others, which we continue

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[9] An additional technical assumption is needed to make this claim strictly true. This observation greatly extands the scope of our approach, enabling it to deal with Tullock contests with general technologies, and with imperfect competition models with imperfect substitutes.

to denote by Q_{-i}, we identify player i's utility-maximizing response. Then, by varying Q_{-i} parametrically, we trace out the graph of player i's best response against the weakest link Q. We consider each of three situations in turn, according to the value of the smallest contribution of players other than i:

1. Suppose that the smallest individual contribution amongst all players excluding player i is zero. Then, under the weakest link model, whatever player i chooses, this places an upper bound of zero on the total level of public good available. The payoff maximizing response by player i must be to choose to contribute zero.
2. Now suppose, instead, that the smallest individual contribution amongst all players excluding player i is some strictly positive level, Q_{-i}, where $Q_{-i} \le Q_i^*$. In this circumstance, player i will want to choose to contribute the quantity $q_i = Q_{-i}$, since up to this value her contribution defines the weakest link and moves her along the 45^o ray through the origin up to her chosen value. However, she will not want to contribute more than Q_{-i}, since the total quantity is constrained at that level.
3. Finally, suppose that the smallest of all other players' contributions is $Q_{-i} > Q_i^*$. In this case, Player i will certainly want to contribute up to the level Q_i^*. Her contribution over this range determines the value of the weakest link. If she were to contribute beyond the level Q_i^*, this would further increase public good provision, but by the definition of Q_i^* this is not in her own interest. Therefore, her best response is to contribute $\hat{q}_i = Q_i^*$.

This reasoning implies a unique pair, $(Q, \hat{q}_i)$, associated with each possible value of $\min\{q_1, q_2, ..., q_{i-1}, q_{i+1}, ... q_n\}$. Figure 4 depicts the graph consisting of all such points. It consists of all the points on the 45^o ray from the origin to the point where $Q = Q_i^*$. No other points could be observed as equilibrium outcomes in a weakest link contribution game in which individual i is a player.

Algebraically, player i's best response is uniquely determined by her replacement function, as follows:

$$\begin{aligned} r_i(Q) &= \min\{q_{-i}\} \text{ if } 0 \le \min\{q_{-i}\} \le Q_i^* \\ r_i(Q) &= Q_i^* \text{ if } \min\{q_{-i}\} > Q_i^* \end{aligned}$$

3.6.2 Nash Equilibrium

Imagine a weakest-link game involving three players. The graph of each player's replacement function is the 45^o ray from the origin to the point of tangency between one of that player's indifference curves and the ray. Suppose that all three have the same preference map and unit cost. Suppose we label players so that if $m_i > m_j$, then $i > j$. Then it follows that $Q_i^* > Q_j^*$. This is the situation shown in Figure 5, where the graph of player i's replacement function is the line segment OQ_i^* and, by the labelling convention, $m_3 > m_2 > m_1$.

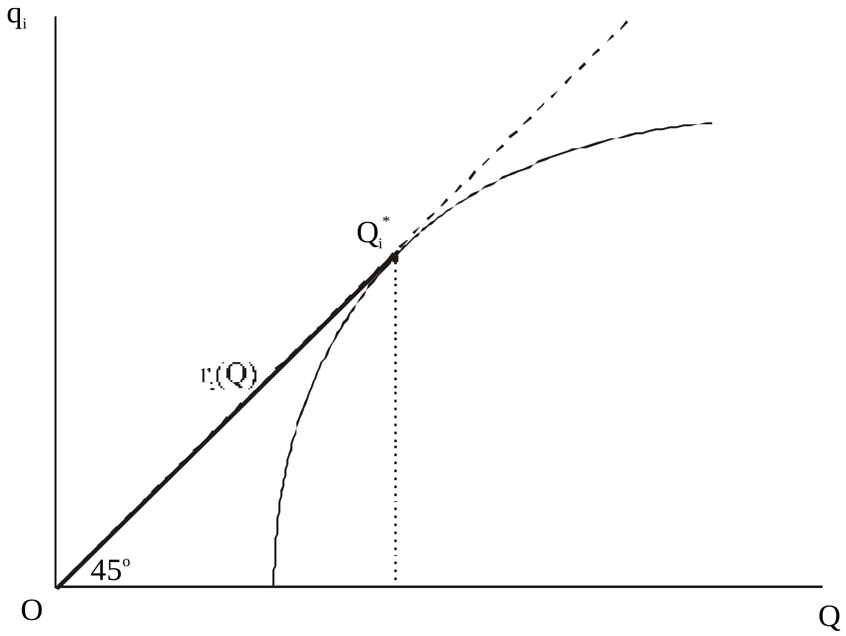

Figure 7:

No level of Q above $\min\{Q_1^*, Q_2^*, Q_3^*\}$ can be consistent with Nash equilibrium. This follows from the observation that player 1 would never find it advantageous to contribute in excess of the quantity Q_1^*. An equilibrium must imply a level of Q consistent with each player choosing a point on the graph of his replacement function. In the example, any value of Q in the interval $[0, Q_1^*]$ is a possible equilibrium, since any such value is consistent with each player choosing his replacement value, and $Q = \min\{r_1(Q), r_2(Q), r_3(Q)\}$, Thus the weaker link model has a continuum of Nash equilibria.

This fact makes it difficult to identify an expected outcome in this game. Let us go along with Hirshleifer's possibly optimistic prediction that, in this situation, players will choose the Pareto dominant allocation amongst all the candidate equilibria. Thus the outcome sees each providing the level Q_1^* in our example. Hirshleifer himself goes on to argue that, in weakest-link situations, underprovision of the public good may not be too serious. However, this is in part a consequence of a degree of symmetry that he has built into our model. Our example suggests that, if preferences, income levels or productivities differ across players, the under-provision at may be very costly to those whose 'ideal' provision is significantly higher than that of the weakest link. It also suggests that a transfer of money income from other players to the weakest link can shift her preferences in (Q, q_i) space in such a way as to increase the resulting equilibrium. Of course, this presupposes that the players are able, as Hirshleifer

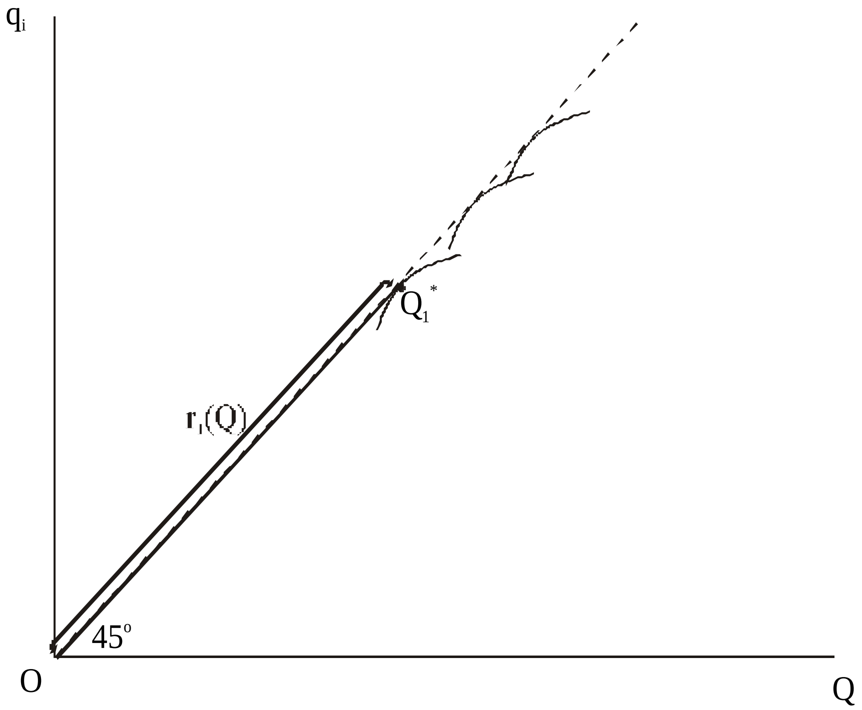

Figure 8:

hoped, to coordinate on the 'best available' equilibrium.

4 Representing Individual Behaviour and Equilibrium in the Best-shot Model

4.1 Individual Behaviour

We follow the same procedure as before, taking the behaviour of other players as parametrically given in order to identify the best response of the typical player. We will find that, in contrast to the other two models, in the best-shot model the best response by player i that is consistent with a given value of the aggregate Q cannot everywhere be uniquely determined. The relationship is a correspondence.

We consider each of three situations in turn, according to the value of the largest contribution of players other than i:

1. Suppose that the largest contribution amongst all players excluding player i, Q_{-i}, lies in the interval $\left[0, \underline{Q}_{-i}\right]$. If player i were to contribute zero,

he would enjoy the quantity of the public good, and his utility would be given by his indifference curve through the point $(Q_{-i}, 0)$ in (Q, q_i) space. This is not utility maximizing, since by choosing the quantity Q^*_i, which would then become the best shot, he could achieve a higher utility level. Hence, all situations in which $0 \leq Q_{-i} < \underline{Q}_{-i}$ uniquely imply the allocation $(Q, q_i) = (Q_{-i}, Q_{-i})$ as the sole allocation consistent with utility maximization by player i.

2. Suppose that $Q_{-i} = \underline{Q}_{-i}$. Then player i is indifferent between taking a free ride and enjoying the bundle $(Q, q_i) = (\underline{Q}_{-i}, 0)$ or choosing to contribute the quantity Q^*_i, in which case he enjoys the bundle $(Q, q_i) = (Q^*_i, Q^*_i)$. Both choices give him the same utility, and cannot be bettered from his viewpoint.

3. Finally, suppose that $Q_{-i} > \underline{Q}_{-i}$. Then player i cannot do better than be a free rider, contributing zero and enjoying the bundle $(Q, q_i) = (\underline{Q}_{-i}, 0)$.

In summary, the points consistent with player i choosing a best response consistent with the choices of others are represented by the line segment $[\underline{Q}_{-i}, m_i]$ along the horizontal axis in Figure 6, together with the single point $(Q, q_i) = (Q^*_i, Q^*_i)$. This defines the graph of player i's replacement correspondence. Note that the total value $Q = Q^*_i$ is consistent with two situations. In one, player i is choosing $\hat{q}_i = Q^*_i$ in response to a situation in which $0 \leq \max\{q_1, q_2, ..., q_{i-1}, q_{i+1}, ..., q_n\} \leq \underline{Q}_{-i}$. Alternatively, the allocation may be one in which player i is choosing $\hat{q}_i = 0$ in response to a situation in which $\max\{q_1, q_2, ..., q_{i-1}, q_{i+1}, ..., q_n\} = Q^*_i$.

4.2 Nash equilibrium

Figure 7 shows the graphs of the replacement correspondences of players in a 3-player best-shot model. The graph of Player i consists of the line segment $[\underline{Q}_{-i}, \infty)$ together with the point Q^*_i.

A Nash equilibrium in this model consists of a level of public good Q in which every player is choosing a value of contribution that lies within her replacement correspondence and, in addition, the aggregate value equals the maximum individual contribution. In the situation depicted in Figure 7, there are three Nash equilibria. At each, player i $[i = 1, 2, 3]$ is choosing the quantity Q^*_i while the other two enjoy free rides, each contributing zero. Such multiplicity of equilibria is a common feature in this model. In the case considered by Hirshleifer, in which players are identical, so that $Q^*_i = Q^*_j$ for all i, j, an n-player game has n Nash equilibria, depending on which individual is the sole contributor. But this is not an inevitable feature of the best-shot model. Sufficient heterogeneity with respect to preferences, income or unit costs may eliminate some of these equilibria. Consider Figure 8.

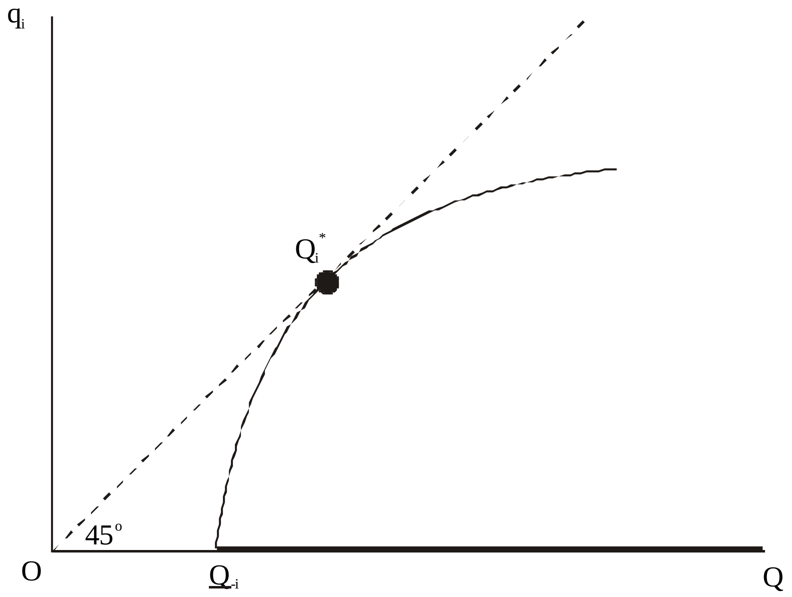

Figure 9:

Here, heterogeneity is such that player 3 would not be content taking a free ride when either of the other players is the sole provider. No total quantity below $\underline{G}_{-3}$ is consistent with equilibrium, and both G_1^* and G_2^* lie below this critical threshold.

This observation suggests two further points. First, the coordination problem that is associated with best-shot provision may not be as serious as may be supposed in a setting in which players are heterogeneous. Furthermore, the possibility of transfers of resources or of technology before the contribution decision is made may, in some situations, help matters by eliminating some of the multiple equilibria.

5 Open Access Resources and Share Functions

5.1 The Basic Open Access Model

Imagine an open access resource, to which we will refer as a fishing ground. The total catch of fish obtained from the ground, X, depends upon the aggregate level L of an input that is applied to the fishing ground. We will think of this as labor. Exploitation of the open access resource is described by a production function $F(L)$ that exhibits diminishing returns to labor. Player i chooses her level of variable input, ℓ_i, taking the input levels of all other players as given. The proportion of total output that is consumed by i, x_i, equals the proportion

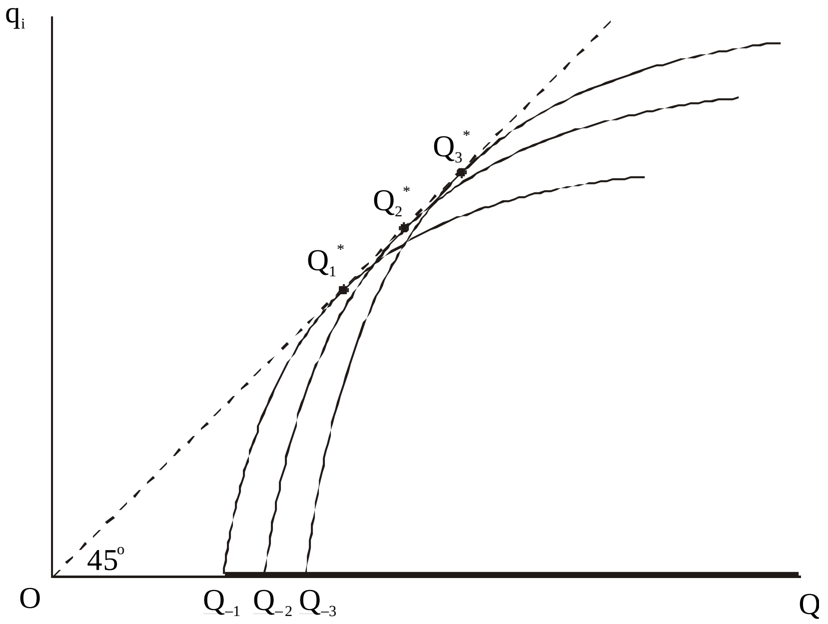

Figure 10:

of total variable input that she supplies: $x_i/X = \ell_i/L$. Player i's preferences are represented by a utility function $u_i(x_i, \ell_i)$.

Players' preferences and the technology are assumed to satisfy the following assumptions:

Preferences $u_i(x_i, l_i)$ is quasiconcave, locally nonsatiable, nondecreasing in x_i, nonincreasing in ℓ_i, continuous and continuously differentiable for x_i, $\ell_i > 0$. Both x_i and ℓ_i are normal.

Technology $F(L)$ is increasing, strictly concave, continuous and continuously differentiable for $L > 0$, and $F(0) = 0$.

A 'boundedness' assumption Either (i) there exists a value of $L > 0$ such that $u_i(F(L), L) = u_i(0,0)$ or (ii) for all $L \geq 0$, $u_i(F(L), L) \leq u_i(0,0)$.

These assumptions are fairly standard, although the last one is often not stated explicitly. We can write player i's utility function as

$$\begin{aligned} u_i(x_i, l_i) &= u_i\left(\left(\frac{\ell_i}{L}\right)F(L), \ell_i\right) \\ &= \nu_i(\ell_i, L). \end{aligned}$$

This game, like the pure public good problem, may be analyzed by deriving and using properties of replacement functions. But there is, at first sight, a

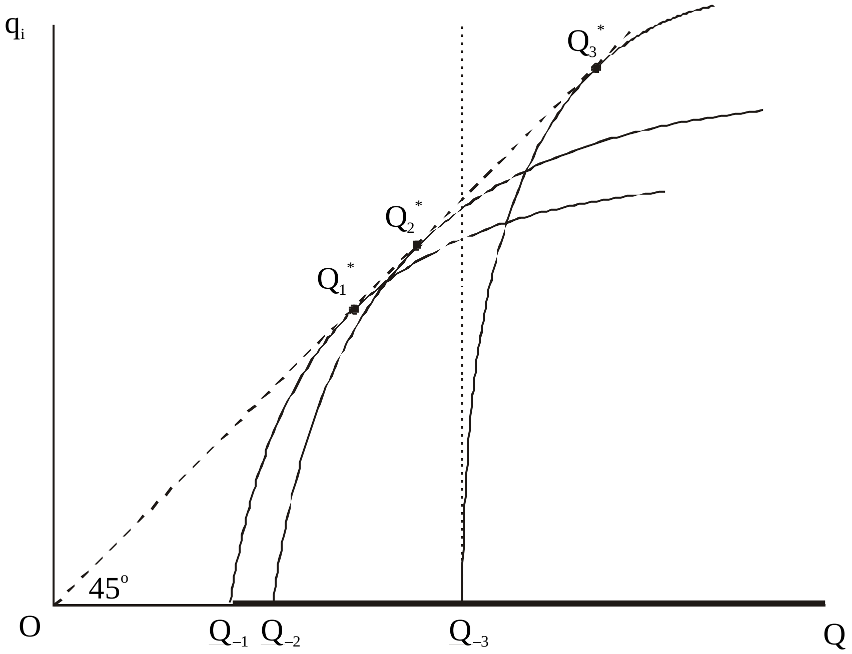

Figure 11:

complication that was absent from the public good model. To see this, consider the following simple numerical example:

Example 1 Let player i's utility function take the form

$$u_i(x_i,\ell_i) = x_i - a_i\ell_i$$

where a_i is an exogenous parameter, and let the aggregate production function take the form

$$X = L^{\frac{1}{2}}.$$

Then, under the proportional sharing rule, player i's utility function may be written as

$$u_i(x_i,\ell_i) = \nu_i(\ell_i, L) = \ell_i L^{-\frac{1}{2}} - a_i\ell_i$$

The first-order condition for an interior solution requires that

$$\frac{\partial \nu_i\left(\hat{\ell}_i, L\right)}{\partial \ell_i} + \frac{\partial \nu_i\left(\hat{\ell}_i, L\right)}{\partial L} = 0$$

or

$$\hat{\ell}_i = 2L\left(1 - a_i\sqrt{L}\right).$$

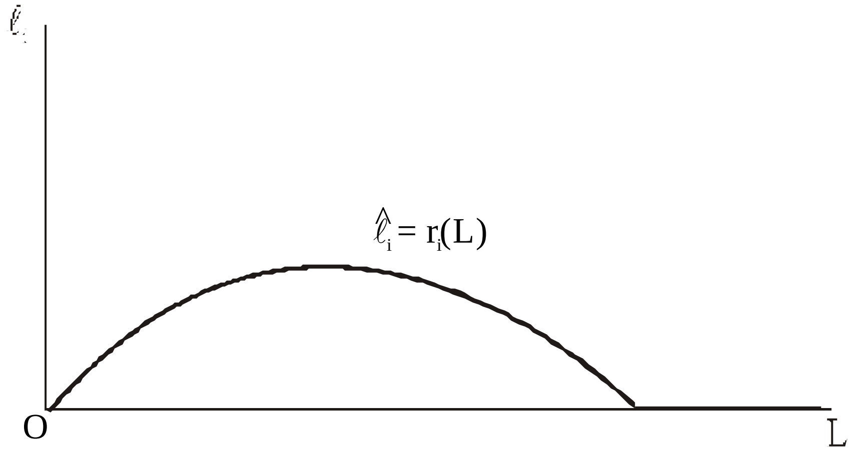

Figure 12:

Taking into account the non-negativity requirement for $\hat\ell_i$, together with the observation that economically meaningful allocations require that $\hat\ell_i \leq L$, player i's replacement function takes the form

$$\hat\ell_i = r_i\left(L : a_i\right) = \min\left\{2L\left(1 - a_i\sqrt{L}\right), 0\ \forall L \geq 1/4a_i^2\right\}.$$

Figure 12 shows the graph of this relationship, and suggests a potential difficulty: Even for this very special example, the replacement function is not monotonic. This immediately raises questions about whether the graph of the aggregate replacement function may snake up and down in such a way as to generate multiple equilibria. I will show that this is not the case. However, although the analysis could all be conducted using replacement functions, I will slightly adapt the replacement function approach in a way that, I feel, produces a more transparent and elegant analysis.

Recall that, when we use replacement functions, a Nash allocation is one at which the following consistency condition is satisfied:

$$r_1(L) + r_2(L) + ... + r_n(L) = L.$$

Divide both sides of this equation by the total[10], L:

$$\frac{r_1(L)}{L} + \frac{r_2(L)}{L} + ... + \frac{r_n(L)}{L} = 1,$$

or

$$s_1(L) + s_2(L) + ... + s_n(L) = 1, \qquad (8)$$

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[10] It may be objected that we risk committing the cardinal sin of dividing by zero. However, as Cornes and Hartley [7] show, this apparent difficulty is easily finessed.

where $s_i(L) \equiv r_i(L)/L$. I call the function $s_i(L)$ player i's share function. It defines her best response, expressed as a share of total effort L, as an explicit function of the aggregate L. The attraction of expressing matters in terms of this function comes from three observations. First, in many applications, $s_i(L)$ can be shown to be monotonic. Moreover, monotonicity of is less restrictive than monotonicity of $r_i(L)$. Finally, monotonicity of every $s_i(L)$ is sufficient for the uniqueness of equilibrium. This is apparent from inspection of the equilibrium requirement as expressed in (8). The RHS is a constant. If every $s_i(L)$ is declining in L for all positive L, then so too is their sum, with the consequence that there can be, at most, a single solution.

We will now dispense with the special forms used in the example, and will show that, in the sharing game with strictly quasiconcave utility functions and a strictly concave production function, individual players have monotonic share functions and there exists a unique noncooperative equilibrium.

5.2 Individual Share Functions

Equilibrium of this model can be analyzed using first principles. First, define player i's marginal rates of substitution and transformation. The first is simply the slope of her indifference curve:

$$MRS_i = -\frac{\partial u_i(x_i, \ell_i)/\partial \ell_i}{\partial u_i(x_i, \ell_i)/\partial x_i}$$

Her marginal rate of transformation is the rate at which labour expended by that player generates consumption available to that player. Under the assumed sharing rule, this is given by

$$MRT_i = \frac{\partial x_i}{\partial \ell_i} = \frac{F(L)}{L} + \frac{\ell_i\left[LF'(L) - F(L)\right]}{L^2}$$

First, note that under the present sharing rule, player i's payoff, MRS_i and MRT_i may all be expressed as functions of the variables $\sigma_i \equiv \ell_i/L$ and the aggregate L itself:

$$\begin{aligned}
u_i\left(\frac{\ell_i}{L}F(L), \ell_i\right) &= u_i(\sigma_i F(L), \sigma_i L) = \nu_i(\sigma_i, L) \\
MRS_i &= -\frac{\partial u_i(.)/\partial \ell_i}{\partial u_i(.)/\partial x_i} = \phi_i(\sigma_i F(L), \sigma_i L) = \zeta_i(\sigma_i, L) \qquad (9) \\
MRT_i &= \frac{F(L)}{L} + \frac{\ell_i\left[LF'(L) - F(L)\right]}{L^2} \\
&= \sigma_i F'(L) + (1 - \sigma_i)\frac{F(L)}{L} = \tau_i(\sigma_i, L) \qquad (10)
\end{aligned}$$

The $\tau_i(.)$ function has a neat form. When player i applies an extra increment of ℓ_i, this raises total output. It also raises her share of total output, since $\frac{\ell_i}{L} = \frac{\ell_i}{\ell_i + L_{-i}}$ also rises. An extra unit of input increases i's consumption of output by an amount that is a weighted sum – indeed, a convex combination – of the overall marginal and average product. Moreover, the weights reflect the player's own significance as an exploiter of the resource.

An interior payoff maximum for player i requires the player to take her input level up to the point at which her marginal benefit, measured by the marginal rate of substitution, equals marginal cost, measured by her marginal rate of transformation:

$$\left[-\frac{\partial u_i\left(.\right)/\partial \ell_i}{\partial u_i\left(.\right)/\partial x_i} = \zeta_i\left(\sigma_i, L\right)\right] = \left[\tau_i\left(\sigma_i, L\right) = \sigma_i F'\left(L\right) + \left(1 - \sigma_i\right)\frac{F\left(L\right)}{L}\right]$$

To extract player i's share function from this expression, we first show that, for a given value of L, the graphs of the functions $\zeta_i\left(\sigma_i, L\right)$ and $\tau_i\left(\sigma_i, L\right)$ have at most a single intersection. Consulting (9), we see that an increase in σ_i alone implies an increase in both x_i and ℓ_i. Our normality assumption implies that $\zeta_i\left(\sigma_i, L\right)$ must rise as a result.

Turning now to (10), an increase in σ_i alone implies a shift of weight from the average to the marginal product term. Our assumptions imply that $F'\left(L\right) < \frac{F(L)}{L}$ for any positive value of L. Thus, the shift in weights towards the smaller of the two terms implies that the value of $\tau_i\left(\sigma_i, L\right)$ falls. Hence, for a given value of L, $\zeta_i\left(\sigma_i, L\right)$ is everywhere increasing in σ_i and $\tau_i\left(\sigma_i, L\right)$ is everywhere decreasing in σ_i. There is therefore, at most, a single value of σ_i for which $\zeta_i\left(\sigma_i, L\right) = \tau_i\left(\sigma_i, L\right)$. Thus we can conclude with the following claim:

Claim 2 There exists a share function for every player in the open access resource game

Inspection of (9) and (10) also reveals the implications of a change in L. An increase in L alone implies an increase in both x_i and ℓ_i. Again, normality implies that, for a given value of σ_i, $\zeta_i\left(\sigma_i, L\right)$ increases. Its graph in Figure 13 shifts upwards. At the same time, an increase in L reduces both the average and marginal product, thereby shifting the graph of τ_i against σ_i downwards.

Hence, the unique point of intersection of the graphs of $\xi_i\left(.\right)$ and $\tau_i\left(.\right)$ in Figure 13 shifts to the left, implying a fall in player i's preferred share. In short, we have

Claim 3 Player i's share function, which we denote by $\sigma_i = s_i\left(L\right)$, is strictly decreasing in L wherever the share value itself is strictly positive.

Nothing in our argument so far rules out the possibility that the share function has an empty domain, has downward jumps, or has a strictly positive limit as $L \to \infty$. Such behaviors would threaten the existence of equilibrium. However, the following property, which is proved in Cornes and Hartley [7], rules out such problematic behavior:

Claim 4 Under our assumptions, for all σ_i such that $0 < \sigma_i \leq 1$, there is a value of L such that $\sigma_i = s_i\left(L\right)$. Furthermore, there is a finite value of L, $\overline{L}_i$, such that $\sigma_i = s_i\left(L\right) = 0$ for all $L \geq \overline{L}_i$.

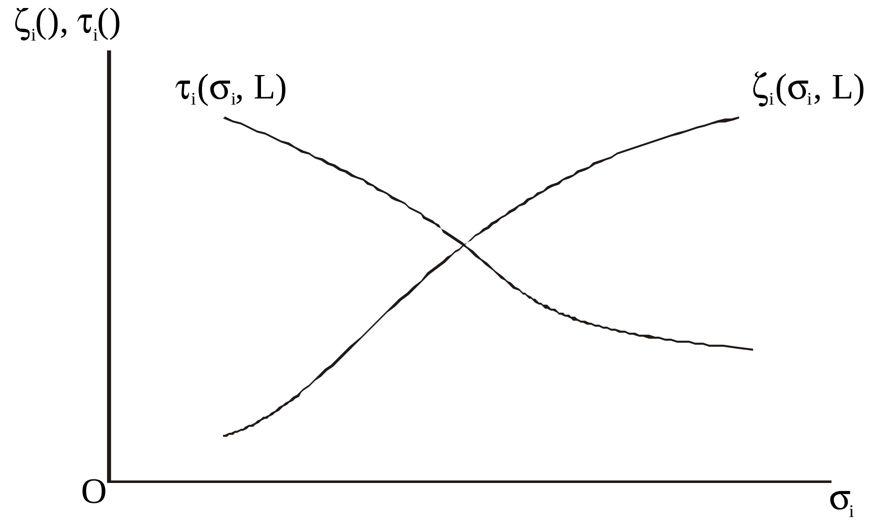

Figure 13:

5.3 Equilibrium and the Aggregate Share Function

We have established the properties of individual share functions in this model. All that remains to do is to define the aggregate share function and to confirm the existence of a unique value of L at which the sum of all share values is unity. This is no more than the expression of the consistency conditions that are the content of the Nash equilibrium notion.

Figure 14 shows the graphs of share functions in a 4-player economy. The thick line shows the graph of the aggregate share function. The figure suggests, correctly, that there must be an equilibrium at some finite value of L, and that there can at most be a single such equilibrium[11]. Perhaps we should state it formally:

Proposition 5 The open access model of this section has a unique Nash equilibrium.

5.4 Some Comparative Statics

Cornes and Hartley [7] show that, in the open access resource model, any shock that leads to a higher equilibrium value of L will leave nonparticipants unaffected, and will reduce the welfare level of participants. It has an ambiguous effect on preferred input levels. Though existing participants will reduce their share of total input, this is consistent with either an increase or a reduction in individual input levels [This simply reflects the fact that share functions are

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[11] A more formal discussion and demonstration of these claims can be found in Cornes and Hartley ??.

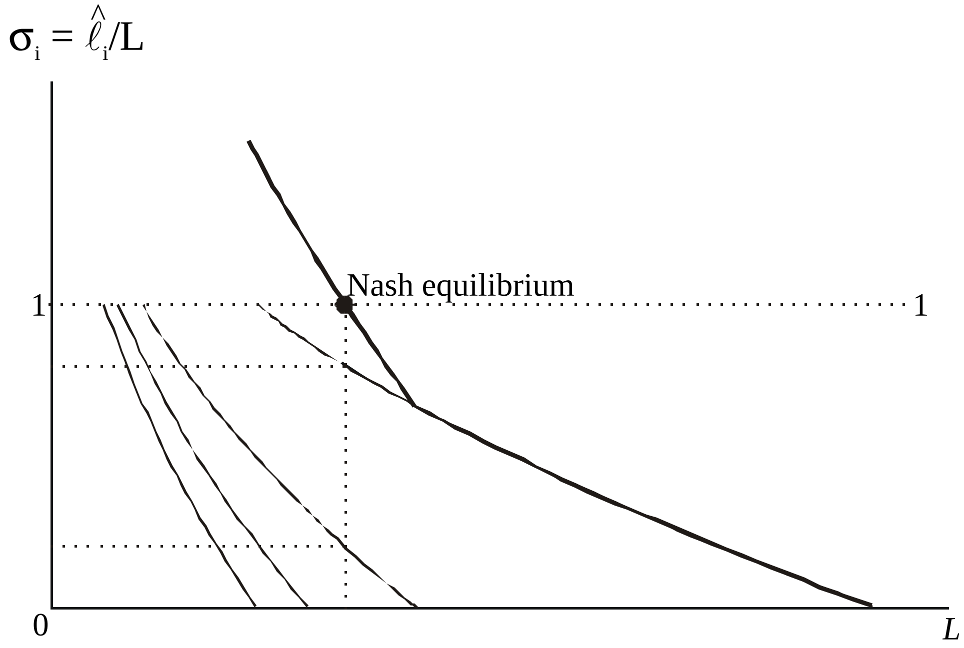

Figure 14:

monotonic, but replacement functions are not]. One can use this set of results to infer the comparative static effects of shocks in which the total level of input, L, is explicitly treated as endogenous - for example, as a result of adding extra players, of introducing a uniform quota, or of individual productivity increases.

6 Other Sharing Rules

6.1 Surplus sharing with Exogenous Shares

The proportional output, or surplus, sharing rule that is associated with the traditional open access resource model is not the only simple sharing rule which our techniques can handle. There are several other well-known sharing rules that lead to an aggregative structure. One such rule is the exogenous sharing rule, under which player i's consumption of the output is given by $x_i = \theta_i F(L)$, where $0 \le \theta_i \le 1$ and $\sum_{j=1}^{n} \theta_j = 1$. This is clearly an aggregative game, since player i's payoff function can be written as $u_i(x_i, \ell_i) = u_i(\theta_i F(L), \ell_i) = \nu_i(\ell_i, L)$. It may therefore be analyzed using the approach suggested in this paper. Moreover, it may be shown that this game also has a unique equilibrium. However, we shall not dwell on this, since we want to suggest a model of which both this and the proportional sharing game are special cases.

6.2 A Mixed Sharing Model

The proportional and exogenous sharing rules are sharply contrasting in one respect. Under the former, there is a clear presumption that there will be over-provision of individual inputs - this is the over-exploitation feature of the tragedy of the commons. Under the latter, there is an equally clear presumption of under-provision. Since each only receives a fraction of her marginal productive, conditions for efficient allocation will not generally be achieved at equilibrium. This is the source of inefficiency associated with underprovision in public good models.

These observations naturally suggest the possibility of combining the two rules in a way that balances the tendency to overprovision in one against the tendency to underprovision in the other. We examine such a sharing rule. First define a 'mixing' parameter λ : by assumption, $0 \leq \lambda \leq 1$. In the fixed sharing rule, player i's share of total output is

$$x_i = \left\{ \lambda \frac{\ell_i}{L} + (1 - \lambda)\theta_i \right\} F(L).$$

In effect, the total output is divided into two piles. The pile consisting of $\lambda F(L)$ is distributed among players in proportion to their input levels, and the rest, $(1 - \lambda) F(L)$, is divided between them according to exogenous shares.

It may be shown, using exactly the same line of reasoning from first principles that we applied to the proportional sharing rule, that the sharing model with the more complicated mixed sharing rule has a unique equilibrium.

Cauley, Cornes and Sandler [5] consider a model of identical individuals and equal exogenous shares. They observe that, if $\lambda = 1$, equilibrium entails overproduction whereas, if $\lambda = 0$, there will be underproduction. They deduce, from continuity, the existence of a value of λ for which the equilibrium is efficient. Cornes and Hartley [8] refine this result by showing that if the mixing parameter is set equal to the equilibrium elasticity of production, the resulting equilibrium is efficient.

If we drop the assumption of equal shares, identical preferences are not essential for an equilibrium allocation to be efficient. For example, when every player's utility function is quasilinear in input, the efficient level of aggregate input is unique. In this case any efficient allocation can be achieved as the equilibrium of a joint production game, provided the mixing parameter is optimal and the exogenous shares are suitably chosen.

6.3 Cost-sharing Models

It is worth remarking that models of cost sharing have essentially the same structure as those of surplus sharing. Consequently, our approach can be applied to the cost-sharing model with minor modifications. To justify this claim, consider the proportional cost sharing game. It has a similar structure to that of the proportional surplus sharing model. To analyse a cost sharing game, it is most convenient to represent the technology by the cost function, $C(X)$, which describes the minimum cost required to produce X units of output - player i's payoff is now a function of output x_i, given by $u_i(x_i, \ell_i) = u_i\left(x_i, \frac{x_i}{X} C(X)\right)$ where $X = \sum_{j=1}^{n} x_j$. The characteristics of the technology are summarized by the following assumptions:

- $C(X)$ is increasing, strictly convex, continuous and continuously differentiable for $X > 0$, and $C(0) = 0$.

We define player i's marginal rates of substitution and of transformation exactly as in the surplus-sharing example. However, their arguments are (σ_i, X), where σ_i is player i's share of total output: $\sigma_i = \frac{x_i}{X}$. Moreover, player i's marginal rate of transformation equals the reciprocal of the convex combination of the marginal and average costs implied by the total output level:

$$\tau_i(x_i, X) = \frac{1}{\sigma_i C'(X) + [1 - \sigma_i] \frac{C(X)}{X}}$$

Comparison with our earlier analysis of the surplus sharing model shows that, where the surplus sharing model generates a convex combination of marginal and average products of a concave production function, the cost sharing model generates the inverse of a convex combination of marginal and average costs associated with a convex cost function. Our analysis of the surplus sharing game, and its diagrammatic representation, with suitable re-labelling of the axes, remain qualitatively valid for the cost-sharing model. Consequently, existence and uniqueness of equilibrium and comparative statics are established along exactly analogous lines.

7 Concluding Comments

I have introduced two tricks - the replacement function, $r_i(Q)$, and the share function, $s_i(Q)$, both of which exploit the special structure of aggregative games in a way that avoids the proliferation of dimensions in games with many heterogeneous players. I have used as my starting point two models - those of public good provision and open access resource exploitation - that lie at the heart of many environmental debates, and to which our tricks can helpfully be applied. Finally, I have indicated a few possible extensions and modifications of these models that should, suggest further applications of these techniques to situations of interest to environmental economists.

There are many ways in which the analysis presented in this paper can be extended. I have already alluded to the fact that the common aggregate about which players care need not be an unweighted sum of individual choices, but may be a more general additively separable function of those choices. Games of incomplete information can also be handled using this framework, as can sequential games. Moreover, it is possible to extend this type of analysis to deal with games in which, instead of just one aggregate, there may be two aggregates that provide a sufficient descriptor of the game. The usefulness to environmental economists of these, and other, extensions remains to be investigated.

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